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4-dimensional Special Orthonormal Frames for Vacuum Gravity

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Abstract

By adopting Nester's 4- dimension al special orthonormal frames, the tetrad equations for vacuum gravity are put into explicitly causal and symmetric hyperbolic form, independent of any time slicing or other gauge or coordinate specializaton.

0.1. Introduction

We have previously given a well set and causal exterior differential system for vacuum gravity, generated by a closed set of differential forms describing the immersion of 4-dimensional spacetime into a flat 10 dimensional space [1]. The orthonormal frame bundle of the latter has a canonical basis of 10 (translation) 1-forms ω^μ and 45 (rotation) 1-forms ω^μ_ν, satisfying the structure equations of the Lie group $ISO(10)$. Dividing the range $\mu = 1, \ldots, 10$ into two ranges $i, j = 1, \cdots, 4$ and $A, B = 5, \cdots, 10$, these are

$$d\omega^i + \omega^i_j \wedge \omega^j + \omega^i_B \wedge \omega^B = 0$$

$$d\omega^A + \omega^A_j \wedge \omega^j + \omega^A_B \wedge \omega^B = 0$$

$$d\omega^i_k + \omega^i_j \wedge \omega^j_k + \omega^i_B \wedge \omega^B_k = 0$$

$$d\omega^i_A + \omega^i_j \wedge \omega^j_A + \omega^i_B \wedge \omega^B_A = 0$$

$$d\omega^A_C + \omega^A_j \wedge \omega^j_C + \omega^A_B \wedge \omega^B_C = 0$$

The exterior differential system is generated by 6 (immersion) 1-forms ω^A, their closure 2-forms $d\omega^A = -\omega^A_i \wedge \omega^i$, and 4 closed 3-forms ensuring Ricci-flatness, namely, $R_{ij} \wedge \omega_k \, \varepsilon^{ijkl}$, where $\frac{1}{2} R^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j = -\omega^i_A \wedge \omega^A_j$ defines the Riemann 2-forms.

The significant result is the calculation of the Cartan characteristic integers

$$s = \{s_0, s_1, s_2, s_3\} = \{6, 6, 10, 8\}.$$

This shows the solutions to be 25 dimensional, regular (i.e., in principle constructed from a nested set of Cauchy-Kowaleski integrations) and causal (i.e., $s_4 = 0$, so the solutions are determined from suitable data set on 3 dimensional immersed manifolds). The solutions are involutory with respect to ω^i, ω^i_j and ω^A_B, i.e., these remain independent when pulled back into a solution manifold and can be adopted as a basis there.

The six basis forms ω^i_j and 15 basis forms ω^A_B, which occur of course in the structure relations, do not appear explicitly in the exterior differential system, showing the solutions to be bundles having 21 dimensional fibers over a four dimensional base. Evidently this expresses arbitrary $O(4)$ (or Lorentzian) rotations of the tetrad frame ω^i and $O(6)$ rotations of the immersion co-frame ω^A at each point of the base. On a four dimensional cross-section all the forms can be expanded on the ω^i basis, the ω^i_j being a metric connection.

In Section 2 we give a new immersion exterior differential system for vacuum gravity that incorporates the higher dimensional special orthonormal frame (11 S01{') conditions proposed by Nester [2] as a generalization of the special orthonormal frame conditions (S01") in three dimensional Riemannian geometry [3] [4] [5]. **Calculation of the Cartan characteristic integers for this exterior differential system for 4 dimensional Riemannian geometry shows that it is also well set and causal, so that, as Nester conjectured, such frames can be imposed without impediment in extended regions of vacuum spacetime.** Solutions are now 19 dimensional, the fibers expressing only arbitrary $O(6)$ rotations of the co-frames.

These exterior differential systems include all integrability conditions for the determination of a metric on the base, or on any 4 dimensional cross section. In terms of base spare coordinates x^μ $(\mu = 1, 2, 3, 4)$, invertible matrices of functions ${}^i\lambda_\mu(x)$ and ${}_i\lambda^\mu(x)$ exist such that ${}^i\lambda_{\mu\ j}\lambda^\mu = \delta^i_j$, $(i = 1, 2, 3, 4)$. We introduce the Minkowski metric, i.e.,

$$\eta^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

to raise and lower left, or Lorentz, indices. Then the metric is given by $g^{\mu\nu} = \eta^{kl}\ {}_k\lambda^\mu\ {}_l\lambda^\nu$. Inserting $\omega^i = {}^i\lambda_\mu\, dx^\mu$ in the structure equations and in the exterior differential system gives the partial differential equations for coordinate components of the tetrad field.

In Section 3 we reformulate the higher dimensional special orthonormal frame system in explicit orthonormal tetrad components, using the signature convention satisfied by η^{ij}. We use the dyadic formalism of references [6] and [7], in which the unit **4-** vector field ${}^4\lambda^\mu$ is given a special meaning: it traces a congruence of timelike world lines, a 3 parameter "fluid" of point observers, to which physical interpretations of the 24 dyadic and 3- vector components of the connection are attributed. The three spacelike unit vectors ${}^a\lambda^\mu$ $(a = 1, 2, 3)$ at each point complete a local orthonormal frame for the observer there. We expand the six 1- forms ω^i_j on the ω^a, ω^4 basis $(a, b = 1, 2, 3)$, and we expand the six 2- forms R^i_j on the $\omega^a \wedge \omega^b$, $\omega^a \wedge \omega^4$ basis subject to the conditions for Ricci-flatness.

The 24 connection components are grouped into the following ensembles: 3 x 3 dyadics K and N, and 3- vectors **a** and ω. The dyadic K has components K_{ab}, and can be resolved into

$$K_{ab} = S_{ab} - \Omega_c\, \varepsilon_{acb},$$

where S_{ab} is the symmetric rate-of-strain 3- tensor Of the observer fluid, and Ω_c is its axial vector of vorticity. or, in convenient notation

$$\mathsf{K} = \mathsf{S} - \mathbf{\Omega} \times \mathsf{I},$$

where I is the unit dyadic. The dyadic N is formed from the nine spacelike Ricci rotation coefficients of the ω^a basis. It has components N_{ab} and can similarly be resolved into a symmetric part and an axial vector n_c:

$$N_{ab} = N_{ab}^{sym} - n_c \varepsilon_{acb}.$$

Again we write in dyadic notation,

$$\mathsf{N} = \mathsf{N}^{sym} - \mathbf{n} \times \mathsf{I}.$$

The vector $\boldsymbol{\omega}$ has components ω_a and is the time-dependent angular velocity of the triad seen by an observer moving along ω^4, with respect to a Fermi-propagated frame. Since w is a standard notation for angular velocity this usage should in context not be difficult to distinguish from the 1- forms ω^a we have used up to this point. (When necessary, we can write the components of an angular velocity as $\bar{\omega}_a$.) The vector **a** has components a_a and is the acceleration of the point observers, i.e., their departure from geodesic motion. The vectors a and $\boldsymbol{\omega}$ are in principle determined operationally by observing spring balances and supported spinning particles in the local frames. The tell components of the Weyl tensor yield symmetric, tracefree dyadics A and B, the so-called electric and magnetic tidal fields. The quantities A, B, K, N, a, and $\boldsymbol{\omega}$ are defined in terms of the ${}_i\lambda_{\mu,\nu}$ in [6].

The dyadic formalism is completed by the use of inner and outer 3--dimensional multiplication (. and x), and by convective derivatives (also known as unit derivations) in the timelike and spacelike directions (" and **D**). Use of 3—covariant spacelike derivation ∇, related to **D** by the 3- connection N, often is more efficient than the use of 1). Detailed exposition of this formalism, including the various 3- vector and dyadic relations, the relations of **1)** and ∇ (involving N), and the 52 general first order dyadic differential equations for vacuum 4- geometry, are to be found in references [6] and ['7].

There are some notational changes of which the reader should be aware. The dyadic we now (and in [7]) denote by N was in [6] denoted by N*. For the dyadic N in [6] (which was symmetric) one should now understand

$$\mathsf{N} - (Tr\,\mathsf{N})\mathsf{I} + \mathbf{n} \times \mathsf{I}$$

and for the vector L in **[6]** one should now write u.

Section **3 also** gives the 12 additional dyadic equations specializing to 4-dimensional special orthonormal frames. The integrability properties of the dyadic equations are briefly discussed there in terms of Cartan's reduced characters and Cartan's test.

In Section 4 we present explicitly the complete set of 64 dyadic equations resulting from substitution of dyadic components in the HSOF exterior differential system of Section 2. **In these equations all time evolution has become explicit.** There are 34 equations for the timelike derivatives of the 24 connection coefficients and the 10 Weyl components, together with an additional 30 transverse equations in which no time derivatives appear. By forming appropriate linear combinations, the final equations have been arranged to show that they have the same first order symmetric hyperbolic (FOSH) structure as recent formalisms involving preferred time slices, designed for application to numerical gravity [8] [9] [10] [11] [12].

A number of known and new "hyperbolic reductions" of the vacuum field equations, and the choices of gauge they allow, have been surveyed by Friedrich [13]. He uses both tetrad formalism, and ADM variables based on preferred time slit.iltg. The rôle of the Bianchi identities is emphasized: the equations for A (in [13], E) and B which propagate as spin-2 massless fields (cf., e.g., [6]), are given in FOSH form. The FOSH equations obtained in the present paper necessarily include these. Nester's conditions 011 the tetrad components of the connection however leave no further freedom in Choice of gauge (or frame), and seem not to have been previously used. **They** result, **in** 1⁴'0S}1 structure **with** constant **coefficients, and force all the** dependent variables to **evolve along null cones.**

An analysis using Cartan's test shows that in the present case the observer fluid necessarily has vorticity, i.e., $\Omega_c \neq 0$, so it is not 3-space orthonormal, and our special orthonormal frames therefore cannot be based therefore on a preferred slicing. Nevertheless, the relative simplicity of these equations may be advantageous. No lapse or shift variables have been introduced. of course, useful related coordinates can still be found. Two are suggested immediately by the exact 2-forms in the exterior differential system, and we note in Appendix] that the dyadic conditions for a harmonic timelike coordinate are again of first order symmetric hyperbolic (FOSH) form, so can simply be added to our results.

A similar application of special orthonormal frames can be made fen" 2 -|] gravity, again leading to a constant coefficient FOSH system. This is outlined in

Appendix 2.

0.20 Higher dimensional special orthonormal frames

Using eight new variables y_i and z_i $(i = 1, 2, 3, 4)$, we prolong the previously given immersion exterior differential system for vacuum gravity, with two additional exact, 2-forms, two additional 3-forms, and their closure 4-forms. The 3-forms and 4-forms essentially define the y_i and z_i in terms of the connection forms (the ω^i_j pulled back into the solutions), and the 2-forms require them to satisfy Rarita-Schwinger equations. When expanded in tetrad components in Section 3 it can be verified that this is precisely Nester's prescription. The generators of this exterior differential system in 63 dimensions are:

$$\omega^A$$

$$\omega^A_i \wedge \omega^i$$

$$(dy_i - \omega^j_i y_j) \wedge \omega^i$$

$$(dz_i - \omega^j_i z_j) \wedge \omega^i$$

$$(\omega^j_i \wedge \omega^k \wedge \omega^l + \tfrac{2}{3} y_i \omega^j \wedge \omega^k \wedge \omega^l)\varepsilon^i_{\cdot jkl}$$

$$\omega_{ij} \wedge \omega^i \wedge \omega^j - \tfrac{1}{3} z_i \omega^j \wedge \omega^k \wedge \omega^l \varepsilon^i_{\cdot jkl}$$

$$\omega^A_j \wedge \omega^A_k \wedge \omega^i \varepsilon_i^{\cdot jkl}$$

$$(-\omega^j_s \wedge \omega^s_i \wedge \omega^k \wedge \omega^l + 2\omega^j_i \wedge \omega^k_s \wedge \omega^s \wedge \omega^l + \tfrac{2}{3} dy_i \wedge \omega^j \wedge \omega^k \wedge \omega^l - 2y_i \omega^j_s \wedge \omega^s \wedge \omega^k \wedge \omega^l)\varepsilon^i_{\cdot jkl}$$

$$-\omega_{is} \wedge \omega^s_j \wedge \omega^i \wedge \omega^j + 2\omega_{ij} \wedge \omega^i_s \wedge \omega^s \wedge \omega^j - (\tfrac{1}{3} dz_i \wedge \omega^j \wedge \omega^k \wedge \omega^l - z_i \omega^j_s \wedge \omega^s \wedge \omega^k \wedge \omega^l)\varepsilon^i_{\cdot jkl} \quad (1)$$

Monte Carlo calculations [1] of the Cartan characteristic integers yield

$$s = \{6, 8, 14, 16\} \ ,$$

therefore the exterior differential system is well set and causal; solutions are 19 dimensional, fibered over 4 dimensions. The ω^A_B do not appear in the exterior differential system, so the fibers express co-frame rotation. But the ω^i_j are explicitly present so frames are specialized and determined up to a simultaneous rotation at every point.

Since the two 2-forms are exact, two further variables can be added, say ζ and η, together with 1-forms

$$d\zeta - y_i \omega^i \ ,$$

$$d\eta - z_i\omega^i.$$

These should be useful for introducing intrinsic coordinates into the HSOF formulation.

0.3. The dyadic components of the connection forms ω^i_j and Riemann forms R^i_j

Now we expand the ω^i_j on ω^i, defining the 24 3-dyadic and 3-vector components **K**, **N**, $\boldsymbol{\omega}$, and a, summarized in the Introduction and described in detail in references [6] and [7]:

$$\begin{aligned}
\omega_{12} &= -\omega_{21} = N_{13}\omega^1 + N_{23}\omega^2 + N_{33}\omega^3 - \bar\omega^3\omega^4 \\
\omega_{23} &= -\omega_{32} = N_{21}\omega^2 + N_{31}\omega^3 + N_{11}\omega^1 - \bar\omega^1\omega^4 \\
\omega_{31} &= -\omega_{13} = N_{32}\omega^3 + N_{12}\omega^1 + N_{22}\omega^2 - \bar\omega^2\omega^4 \\
\omega_{14} &= -\omega_{41} = K_{11}\omega^1 + K_{12}\omega^2 + K_{13}\omega^3 + a^1\omega^4 \\
\omega_{24} &= -\omega_{42} = K_{22}\omega^2 + K_{23}\omega^3 + K_{21}\omega^1 + a^2\omega^4 \\
\omega_{34} &= -\omega_{43} = K_{33}\omega^3 + K_{31}\omega^1 + K_{32}\omega^2 + a^3\omega^4
\end{aligned} \tag{2}$$

We also expand the Riemann 2-forms on a basis consisting of $\omega^a \wedge \omega^4$ and $\omega^a\wedge\omega^b\,(a,b=1,2,3)$, to define (in the Ricci-flat case) symmetric, trace free, Weyl dyadics A and B:

$$\begin{aligned}
R_{12} &= -R_{21} = B_{31}\omega^1\wedge\omega^4 + B_{32}\omega^2\wedge\omega^4 + B_{33}\omega^3\wedge\omega^4 + A_{31}\omega^2\wedge\omega^3 + A_{32}\omega^3\wedge\omega^1 + A_{33}\omega^1\wedge\omega^2 \\
R_{23} &= -R_{32} = B_{12}\omega^2\wedge\omega^4 + B_{13}\omega^3\wedge\omega^4 + B_{11}\omega^1\wedge\omega^4 + A_{12}\omega^3\wedge\omega^1 + A_{13}\omega^1\wedge\omega^2 + A_{11}\omega^2\wedge\omega^3 \\
R_{31} &= -R_{13} = B_{23}\omega^3\wedge\omega^4 + B_{21}\omega^1\wedge\omega^4 + B_{22}\omega^2\wedge\omega^4 + A_{23}\omega^1\wedge\omega^2 + A_{21}\omega^2\wedge\omega^3 + A_{22}\omega^3\wedge\omega^1 \\
R_{14} &= -R_{41} = A_{11}\omega^1\wedge\omega^4 + A_{12}\omega^2\wedge\omega^4 + A_{13}\omega^3\wedge\omega^4 + B_{11}\omega^2\wedge\omega^3 + B_{12}\omega^3\wedge\omega^1 + B_{13}\omega^1\wedge\omega^2 \\
R_{24} &= -R_{42} = A_{22}\omega^2\wedge\omega^4 + A_{23}\omega^3\wedge\omega^4 + A_{21}\omega^1\wedge\omega^4 + B_{22}\omega^3\wedge\omega^1 + B_{23}\omega^1\wedge\omega^2 + B_{21}\omega^2\wedge\omega^3 \\
R_{34} &= -R_{43} = A_{33}\omega^3\wedge\omega^4 + A_{31}\omega^1\wedge\omega^4 + A_{32}\omega^2\wedge\omega^4 + B_{33}\omega^1\wedge\omega^2 + B_{31}\omega^2\wedge\omega^3 + B_{32}\omega^3\wedge\omega^1
\end{aligned} \tag{3}$$

The 10 Weyl components satisfy $A_{ab} = A_{ba}$, $A_{aa} = 0$, $B_{ab} = B_{ba}$, and $B_{aa} = 0$. It can be verified that the Riemann symmetries are satisfied, namely,

$$R^a_b \wedge \omega^b + R^a_4 \wedge \omega^4 = 0$$

and

$$R^4_b \wedge \omega^b = 0,$$

and also that the four 3- form conditions for Ricci-flatness, namely,

$$R^i_j \wedge \omega^k \varepsilon^j_{\cdot ikl} = 0$$

have been imposed.

From the 3- forms of (1) we can now find the component expansions of the fields y_i and z_i, on an integral manifold of the exterior differential system. These are given by the following:

$$y_i = (y^a, - y^4) = (\mathsf{N}\dot{\times}\mathsf{I} - \mathbf{a}, - Tr\,\mathsf{K}) \tag{4}$$

$$z_i = (z^a, - z^4) = (\mathsf{K}\dot{\times}\mathsf{I} - \omega, Tr\,\mathsf{N}). \tag{5}$$

We could alternatively have used vectors $2\mathbf{n} = \mathsf{N}\dot{\times}\mathsf{I}$ and $2\Omega = \mathsf{K}\dot{\times}\mathsf{I}$. The twelve dyadic first order equations arising from the two additional HSOF 2- forms then are:

$$\nabla(Tr\,\mathsf{K}) + (Tr\,\mathsf{K})\mathbf{a} + (\mathsf{N}\dot{\times}\mathsf{I} - \mathbf{a})" - \{ \mathsf{K}\cdot(\mathsf{N}\dot{\times}\mathsf{I} - \mathbf{a}) + \omega \times (\mathsf{N}\dot{\times}\mathsf{I} - \mathbf{a}) = 0 \tag{6}$$

$$\nabla \times (\mathsf{N}\dot{\times}\mathsf{I} - \mathbf{a}) + (Tr\,\mathsf{K})\mathsf{K}\dot{\times}\mathsf{I} = 0 \tag{7}$$

$$\nabla(Tr\,\mathsf{N}) + (Tr\,\mathsf{N})\,\mathbf{a} - (\mathsf{K}\dot{\times}\mathsf{I} - \omega)\cdot\mathsf{K}\cdot(\mathsf{K}\dot{\times}\mathsf{I} - \omega) - \omega \times (\mathsf{K}\dot{\times}\mathsf{I}) = 0 \tag{8}$$

$$\nabla \times (\mathsf{K}\dot{\times}\mathsf{I} - \omega) - (Tr\cdot\mathsf{N})\mathsf{K}\dot{\times}\mathsf{I} = 0. \tag{9}$$

The sets of first order equations we have derived, where the dyadic components are taken as dependent variables and the ω^i form an independent basis set, may be analyzed alternatively by another of Cartan's techniques. **We must know that the set includes all integrability conditions**, and it must also be possible to write an exterior differential system such that the left, hand sides **linearly** involve exterior derivatives of the dependent variables (e.g., no terms of the form $dK_{ab} \wedge dN_{cd}$), while the right hand sides only involve forms in the adopted independent basis (here ω^i, $\omega^i \wedge \omega^j$, etc.). So-called **reduced** characters s'_i are then conveniently computed from the left-hand sides alone, and **Cartan's test** is to calculate

$$h = \sum_{i=0}^{g-1}(g - i)\cdot s'_i .$$

If' h is equal to the number of independent first order equations, one has established involutivity, the well set nature of the problem. Moreover, if

$$s'_g = n - g - \sum_{i=0}^{g-1} s'_i = 0,$$

the Cauchy-Kowalcski solutions are unique and causal.

The 52 general vacuum dyadic equations [6] [14] were of this form, having six 2- forms and six 3– forms in 34 dependent variables. Their left hand sides are:

$$dN_{ab} \wedge \omega^b - d\omega_a \wedge \omega^4$$

$$dK_{ab} \wedge \omega^b + da_a \wedge \omega^4$$

$$dA_{ab} \wedge \omega^b \wedge \omega^4 - \tfrac{1}{6}\, dB_{ab} \wedge \omega^c \wedge \omega^d \varepsilon^b_{.cd}$$

$$dB_{ab} \wedge \omega^b + \tfrac{1}{6}\, dA_{ab} \wedge \omega^c \wedge \omega^d \varepsilon^b_{.cd}.$$

We calculate s' = {O, 6, 12, 1 O} , so $h = 52$. There are however $s'_4 = 6$ arbitrary functions in the solution. The new HSOF system of dyadic equations adjoins 12 equations in two additional 2- forms to express equations (6)- (9). Their left hand sides arc:

$$d(2n_a - a_a) \wedge \omega^a + d(Tr\,\mathsf{K}) \wedge \omega^4$$

$$d(2\Omega_a - \omega_a) \wedge \omega^a - d(Tr\,\mathsf{N}) \wedge \omega^4.$$

The reduced characters arc s' = {0,8, 14, 12} , $h = 64$ and $s'_4 = 0$.

0.4. The FOSH dyadic equations

Linear combinations of the 52 general dyadic equations and the 12 new conditions due to Nester can now easily be made to put the result in FOSH form. The result is 34 equations involving the time derivatives and symmetric space derivatives of the dyadic variables and **A** and **B**, and **30** "constraint" or transverse relations not involving time derivatives. We give them in the following, written in full with their right hand sides:

$$\dot{\mathbf{a}} - \nabla \cdot \mathsf{K}^T + \nabla \times \boldsymbol{\omega} = - \mathsf{K} \dot{\times} \mathsf{N} - \boldsymbol{\omega} \cdot \mathsf{N} + (Tr\,\mathsf{N})\boldsymbol{\omega} + 2\mathsf{K} \cdot \mathbf{n} \qquad (10)$$

$$\dot{\mathbf{w}} + \nabla \cdot \mathsf{N}^T - \nabla \times \mathbf{a} = - (Tr\,\mathsf{N})\mathbf{a} + (2\boldsymbol{\Omega} - \boldsymbol{\omega}) \cdot \mathsf{K} - 2(Tr\,\mathsf{K})\,\boldsymbol{\Omega} - 2\mathbf{n} \cdot \mathsf{N} \qquad (11)$$

$$\dot{\mathsf{K}} - \mathbf{a}\nabla - 2\nabla\mathbf{n} + 2\mathbf{n}\nabla = - \mathsf{K} \cdot \mathsf{K} - 2(Tr\,\mathsf{K})\boldsymbol{\Omega} \times \mathsf{I} - \boldsymbol{\omega} \times \mathsf{K} + \mathsf{K} \times \boldsymbol{\omega} + \mathbf{a}\mathbf{a} - \mathsf{A} \qquad (12)$$

$$\dot{\mathsf{N}} + \omega\nabla + 2\nabla\boldsymbol{\Omega} - 2\boldsymbol{\Omega}\nabla = -\mathsf{K}\cdot\mathsf{N} - \omega\times\mathsf{N} + \mathsf{K}\times\mathbf{a} - \mathbf{a}\omega - 2(Tr\,\mathsf{N})\boldsymbol{\Omega}\times\mathsf{I} + \mathsf{B}. \quad (13)$$

We mention in passing that using $\mathbf{D}$ instead of ∇ in general makes the right hand sides of these equations less concise. The prominent exception to this statement is the first equation, which becomes homogeneous and linear:

$$\dot{\mathbf{a}} - \mathbf{D}\cdot\mathsf{K}^T + \mathbf{D}\times\omega = 0$$

Here it has been convenient to use n and Ω, and ∇ can operate from the right as well as from the left, to express transposed indices. It is best to return to the nine components of K and of N to see the FOSH structure; e.g. write $(2\nabla\mathbf{n})_{11} = \nabla_1 N_{23} - \nabla_1 N_{32}$, $(2\mathbf{n}\nabla)_{23} = \nabla_3 N_{31} - \nabla_3 N_{13}$, $(2\boldsymbol{\Omega}\nabla)_{22} = \nabla_2 K_{31} - \nabla_2 K_{13}$, etc. The left hand sides of the equations are a linear operator shown in Figure 1.

The 24 constraint equations not containing time derivatives are:

$$\nabla\times\mathsf{K} = 2\boldsymbol{\Omega}\mathbf{a} - \mathsf{B} \quad (14)$$

$$\nabla\times\mathsf{N} = -\tfrac{1}{2}\mathsf{N}^T \overset{\times}{\underset{\times}{}} \mathsf{N} - \tfrac{1}{2}\mathsf{K} \overset{\times}{\underset{\times}{}} \mathsf{K} + (\boldsymbol{\Omega}\cdot\mathsf{K})\times\mathsf{I} - 2\boldsymbol{\Omega}\omega - \mathsf{A} \quad (15)$$

$$\nabla\times(\mathbf{a} - 2\boldsymbol{\Omega}) = 2(Tr\,\mathsf{K})\boldsymbol{\Omega} \quad (16)$$

$$\nabla\times(\omega - 2\boldsymbol{\Omega}) = -2(Tr\,\mathsf{N})\boldsymbol{\Omega} \quad (17)$$

By taking a second time derivative of the FOSH equations, permuting space and time derivatives, and substituting back both the FOSH and the constraint equations, it can be seen that the dyadic variables all propagate causally along null cones.

The FOSH equations for the Weyl components are those for traceless transverse massless spin-2 fields (dyadic and Bianchi identities, linearly combined)

$$2\dot{\mathsf{B}} - \nabla\times\mathsf{A} + \mathsf{A}\times\nabla =$$

$$\mathsf{B}\times\omega - \omega\times\mathsf{B} - \mathsf{A}\times\mathbf{a} + \mathbf{a}\times\mathsf{A} - \mathsf{K}^T\cdot\mathsf{B} + \mathsf{B}\cdot\mathsf{K} - 2(Tr\,\mathsf{K})\mathsf{B} + \mathsf{K} \overset{\times}{\underset{\times}{}} \mathsf{B} + \mathsf{B} \overset{\times}{\underset{\times}{}} \mathsf{K} \quad (18)$$

$$2\dot{\mathsf{A}} + \nabla\times\mathsf{B} - \mathsf{B}\times\nabla =$$

$$\mathsf{A}\times\omega - \omega\times\mathsf{A} + \mathsf{B}\times\mathbf{a} - \mathbf{a}\times\mathsf{B} + \mathsf{K}^T\cdot\mathsf{A} + \mathsf{A}\cdot\mathsf{K} - 2(Tr\,\mathsf{K})\mathsf{A} + \mathsf{K} \overset{\times}{\underset{\times}{}} \mathsf{A} + \mathsf{A} \overset{\times}{\underset{\times}{}} \mathsf{K} \quad (19)$$

with constraint equations

$$\nabla \cdot \mathsf{A} = - \mathsf{K} \dot{\times} \mathsf{B} - 4\boldsymbol{\Omega} \cdot \mathsf{B} \tag{20}$$

$$\nabla \cdot \mathbf{B} = \mathsf{K} \dot{\times} \mathsf{A} + 4\boldsymbol{\Omega} \cdot \mathsf{A}. \tag{21}$$

The left hand sides of the Bianchi equations are the linear operator given in Figure 2.

Finally, it should be remarked that our derivations from well set exterior differential systems obviate any need to verify that the transverse constraint equations (not involving time derivatives) are compatible with the FOSH system, and that they are propagated invariantly by it.

0.5. Appendix 1: Harmonic and co-moving coordinates

The formulation of this paper is coordinate-free and gauge independent. It may however also be of use to briefly record how a harmonic time coordinate and co-moving (with ${}_4\lambda^\mu$) spacelike coordinates can be adopted.

If in coordinate language $t_{;\mu\nu}\, g^{\mu\nu} = 0$ $(\mu, \nu = 1, \ldots, 4)$, the tetrad version, introducing fields $\phi = \dot{t}, \mathbf{A} = \nabla t$, is

$$\dot{\mathbf{A}} - \nabla\phi = -\mathsf{K} \cdot \mathbf{A} - \omega \times \mathbf{A} + \phi\mathbf{a}$$

$$\dot{\phi} - \nabla \cdot \mathbf{A} = \mathbf{a} \cdot \mathbf{A} - \phi(Tr\ \mathsf{K})$$

$$\nabla \times \mathbf{A} = 2\phi\Omega$$

This set consists of four FOSH equations plus 3 transverse constraints; it can be added to, and solved simultaneously with, the equations in %c.tie]) 4.

Co-moving spacelike coordinates, say x^α $(\alpha = 1, \ldots, 3)$, such that $\dot{x}^\alpha = 0$ are found by setting $\mathbf{e}^\alpha = \nabla x^\alpha$, and the integrability conditions for this are

$$\dot{\mathbf{e}}^\alpha = -\mathsf{K} \cdot \mathbf{e}^\alpha - \omega \times \mathbf{e}^\alpha$$

$$\nabla \times \mathbf{e}^\alpha = 0.$$

The $\mathbf{e}^\alpha$ are, understandably, the first dependent variables we have found whose causal propagation is strictly timelike and not along the local null cone. We introduce coordinate components $\mathbf{A} \cdot \mathbf{e}^\alpha = A^\alpha$ and $h^{\alpha\beta} = \mathbf{e}^\alpha\ \mathbf{e}^\beta$, and also calculate the inverse $h_{\alpha\beta}$ ($h^{\alpha\gamma} h_{\alpha\beta} = \delta^\gamma_\beta$). Then ϕ, $A_\alpha = h_{\alpha\beta}A^\beta$ and $h_{\alpha\beta}$, functions of x^α, t, enter the final line element:

$$ds^2 = -\phi^{-2}\,dt^2 + 2\phi^{-2}A_\alpha\,dx^\alpha dt + (h_{\alpha\beta} - \phi^{-2}A_\alpha A_\beta)\,dx^\alpha\,dx^\beta$$

In covariant 4 – vector terms, the coordinate components of the tetrad vectors take the form

$${}_r\lambda^\mu = \begin{pmatrix} {}_a\lambda^\alpha & {}_aA \\ 0 & \phi \end{pmatrix}$$

$${}^r\lambda_\mu = \begin{pmatrix} {}^a\lambda_\alpha & 0 \\ -\phi^{-1}A_\alpha & \phi^{-1} \end{pmatrix}$$

The orthonormal triad components of the vector A are written ${}_a\lambda^\alpha\ A_{,r}$.

0.6. Appendix 2: 2 + 1 dimensional gravity

An entirely parallel development can be made of special orthonormal frames (SOF) in 2+1 gravity. The exterior differential system for immersion of 3- dimensional flat spaces in the 21 dimensional orthonormal frame bundle over 6 dimensional flat space, which is generated by the usual immersion forms w^A and $d\omega^A$ $(A = 4,5,6)$, and the Riemann 2- forms R^a_b $(a, b, c = 1,2,3)$, has $s = \{3,6,3\}$ and $g = 9$. We add a 2- form

$$\omega^b_a \wedge \omega^c \varepsilon^a_{.bc} + y_a\omega^b \wedge \omega^c \varepsilon^a_{.bc},$$

a 3- form

$$\omega_{ab} \wedge \omega^a \wedge \omega^b - z\omega^1 \wedge \omega^2 \wedge \omega^3$$

and their closures, to introduce 4 new variables y_a, and z, so the exterior differential system is prolonged to a total of 25 dimensions. The SOF conditions are imposed as two additional exact forms (which prolong, or adjoin, solutions of a linear Dirac equation [4]), namely,

$$dz$$

$$d(y_a\omega^a).$$

Now the Cartan characters show the exterior differential system to be well set and causal with

$$s = \{4,8,7\}\ ,$$

and $g = 6$. Solutions are 3- parameter co frame rotation bundles over 3- space, determined by causal integration from 2- spaces.

Orthonormal triad formalism for 3- geometry yields 9 equations for 9 Ricci rotation coefficients grouped in a dyadic N:

$$\nabla \times \mathsf{N} + \tfrac{1}{2}\mathsf{N} \mathop{\times}\limits^{\times} \mathsf{N} = 0$$

The SOF conditions require the identification of 2 as Tr N and the y_a as the components of n = $\frac{1}{2}$N xl. To the nine equations for a flat 3-- manifold are added six SOF equations:

$$\nabla(Tr\,\mathsf{N}) = 0$$

$$\nabla \times \mathbf{n} = 0.$$

Inserting a −1 corresponding to timelike signature of the 3 direction, the complete set, falls into 1"OS11 form with 6 transverse equations. The left hand side of the evolution operator is given in Figure 3.

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$$
\begin{pmatrix}
D_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -D_2 & D_1 & -D_1 & \cdot & D_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_2 & \cdot & \cdot \\
\cdot & D_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_2 & \cdot & -D_3 & D_3 & -D_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_3 & \cdot \\
\cdot & \cdot & D_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -D_1 & D_3 & \cdot & \cdot & D_1 & -D_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_1 \\
\cdot & \cdot & \cdot & D_4 & \cdot & \cdot & \cdot & \cdot & \cdot & D_1 & -D_3 & \cdot & \cdot & -D_1 & D_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & D_4 & \cdot & \cdot & \cdot & \cdot & \cdot & D_2 & -D_1 & D_1 & \cdot & -D_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & D_4 & \cdot & \cdot & \cdot & -D_2 & \cdot & D_3 & -D_3 & D_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_3 \\
\cdot & D_2 & -D_1 & D_1 & \cdot & -D_2 & \cdot & \cdot & \cdot & D_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -D_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
-D_2 & \cdot & D_3 & -D_3 & D_2 & \cdot & \cdot & \cdot & \cdot & \cdot & D_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -D_3 & \cdot & \cdot & \cdot & \cdot \\
D_1 & -D_3 & \cdot & \cdot & -D_1 & D_3 & \cdot & \cdot & \cdot & \cdot & \cdot & D_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -D_1 & \cdot & \cdot & \cdot \\
-D_1 & D_3 & \cdot & \cdot & D_1 & -D_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_4 & \cdot & \cdot & \cdot & \cdot & \cdot & -D_3 & \cdot & \cdot & \cdot & \cdot & \cdot \\
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D_2 & \cdot & -D_3 & D_3 & -D_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_4 & \cdot & \cdot & \cdot & \cdot & \cdot & -D_2 & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -D_2 & \cdot & \cdot & -D_3 & \cdot & \cdot & -D_1 & \cdot & \cdot & D_4 & \cdot & \cdot & \cdot & -D_3 & D_2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -D_3 & \cdot & \cdot & -D_1 & \cdot & \cdot & -D_2 & \cdot & \cdot & D_4 & \cdot & D_3 & \cdot & -D_1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -D_1 & \cdot & \cdot & -D_2 & \cdot & \cdot & -D_3 & \cdot & \cdot & D_4 & -D_2 & D_1 & \cdot \\
D_2 & \cdot & \cdot & D_3 & \cdot & \cdot & D_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_3 & -D_2 & D_4 & \cdot & \cdot \\
\cdot & D_3 & \cdot & \cdot & D_1 & \cdot & \cdot & D_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -D_3 & \cdot & D_1 & \cdot & D_4 & \cdot \\
\cdot & \cdot & D_1 & \cdot & \cdot & D_2 & \cdot & \cdot & D_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_2 & -D_1 & \cdot & \cdot & \cdot & D_4
\end{pmatrix}
\begin{pmatrix}
N_{12} \\ N_{23} \\ N_{31} \\ N_{13} \\ N_{21} \\ N_{32} \\ N_{11} \\ N_{22} \\ N_{33} \\ K_{12} \\ K_{23} \\ K_{31} \\ K_{13} \\ K_{21} \\ K_{32} \\ K_{11} \\ K_{22} \\ K_{33} \\ a_1 \\ a_2 \\ a_3 \\ \omega_1 \\ \omega_2 \\ \omega_3
\end{pmatrix}
$$

Figure 1: The FOSH operator shown acting on the connection coefficients. Dot indicates zero.

$$
\begin{bmatrix}
D_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -D_3 & \cdot & \cdot & D_2 \\
\cdot & D_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_3 & -D_1 & \cdot \\
\cdot & \cdot & D_4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_1 & -D_2 \\
\cdot & \cdot & \cdot & 2D_4 & \cdot & \cdot & D_3 & -D_3 & \cdot & \cdot & D_2 & -D_1 \\
\cdot & \cdot & \cdot & \cdot & 2D_4 & \cdot & \cdot & D_1 & -D_1 & -D_2 & \cdot & D_3 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 2D_4 & -D_2 & \cdot & D_2 & D_1 & -D_3 & \cdot \\
\cdot & \cdot & \cdot & D_3 & \cdot & -D_2 & D_4 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -D_3 & D_1 & \cdot & \cdot & D_4 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -D_1 & D_2 & \cdot & \cdot & D_4 & \cdot & \cdot & \cdot \\
-D_3 & D_3 & \cdot & \cdot & -D_2 & D_1 & \cdot & \cdot & \cdot & 2D_4 & \cdot & \cdot \\
\cdot & -D_1 & D_1 & D_2 & \cdot & -D_3 & \cdot & \cdot & \cdot & \cdot & 2D_4 & \cdot \\
D_2 & \cdot & -D_2 & -D_1 & D_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2D_4
\end{bmatrix}
\begin{bmatrix}
A_{11} \\ A_{22} \\ A_{33} \\ A_{12} \\ A_{23} \\ A_{31} \\ B_{11} \\ B_{22} \\ B_{33} \\ B_{12} \\ B_{23} \\ B_{31}
\end{bmatrix}
$$

Figure 2: The FOSH operator shown acting on the Weyl components.

$$
\begin{bmatrix}
D_3 & -D_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-D_1 & D_3 & -D_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -D_2 & D_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & D_3 & -D_1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -D_1 & D_3 & -D_2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -D_2 & D_3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & D_3 & -D_1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -D_1 & D_3 & -D_2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -D_2 & D_3
\end{bmatrix}
\begin{bmatrix}
N_{11} \\ N_{31} \\ N_{21} \\ N_{12} \\ N_{32} \\ N_{22} \\ N_{13} \\ N_{33} \\ N_{23}
\end{bmatrix}
$$

Figure 3: The FOSH operator in 2+1 gravity shown acting on the components N_{ab}.